Observability of a large control vector in a 4D-Var ocean data assimilation

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- Background
- 4D-Var
- Observability matrix in 4D-Var
- Algorithm
- Test of the algorithm with QG model
- Summary

GOAPP
Global Ocean-Atmosphere Prediction and Predictability

Fisheries and Oceans Canada Pêches et Océans Canada

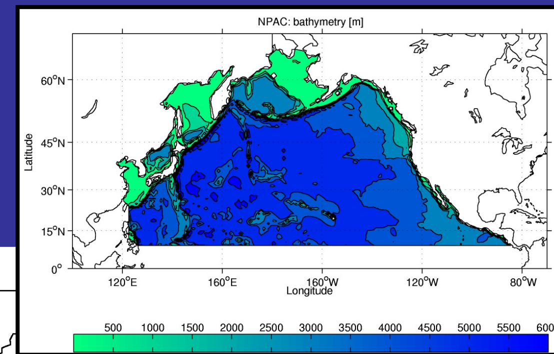
Canadian Foundation for Climate and Atmospheric Sciences (CFCAS)
Fondation canadienne pour les sciences du climat et de l'atmosphère (FCSCA)

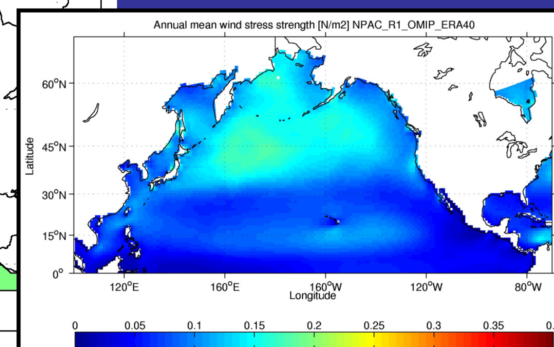
CMOS 2009, Halifax NS

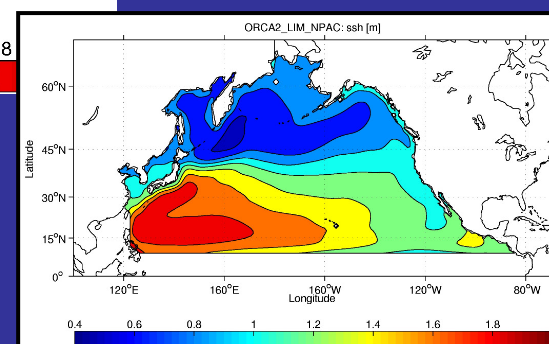
Background

forward modeling

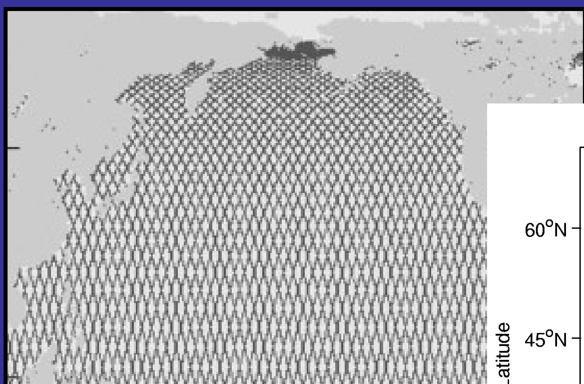
control

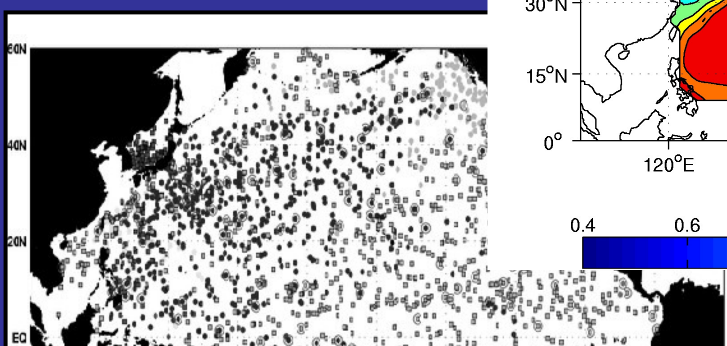




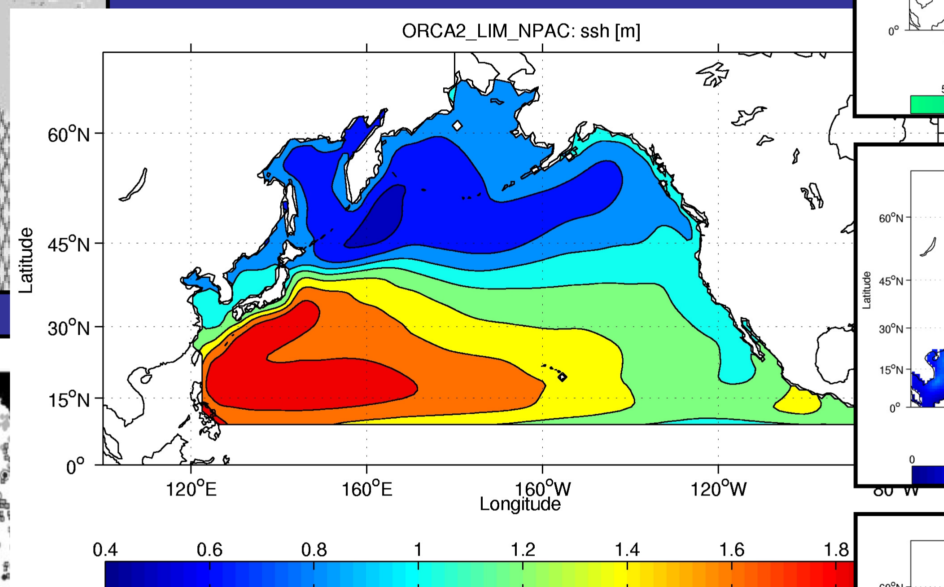


data



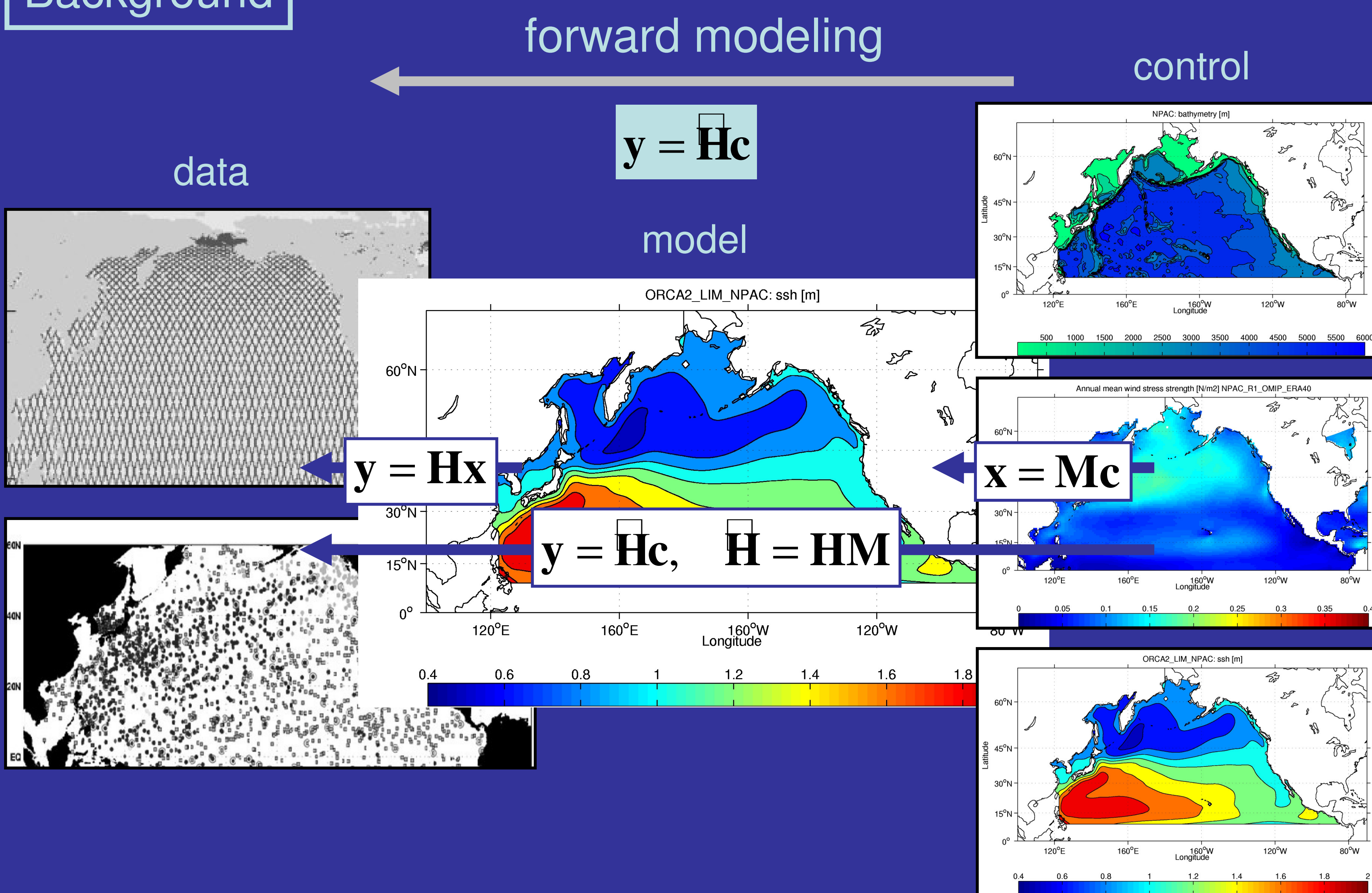


model



$$\mathbf{y} = \mathbf{Hc}$$

Background

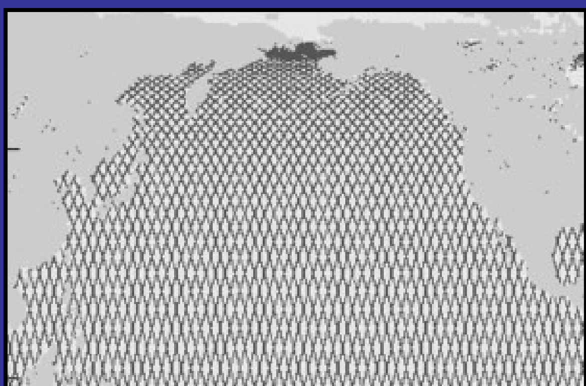

forward modeling
control
y = Hc
data
model
y = Hx
x = Mc
y = Hc,   H = HM
ORCA2_LIM_NPAC: ssh [m]
Longitude
NPAC: bathymetry [m]
Annual mean wind stress strength [N/m2] NPAC_R1_OMIP_ERA40

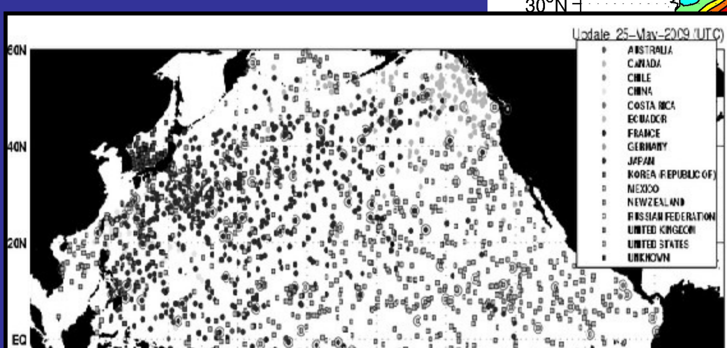
Background

inverse modeling

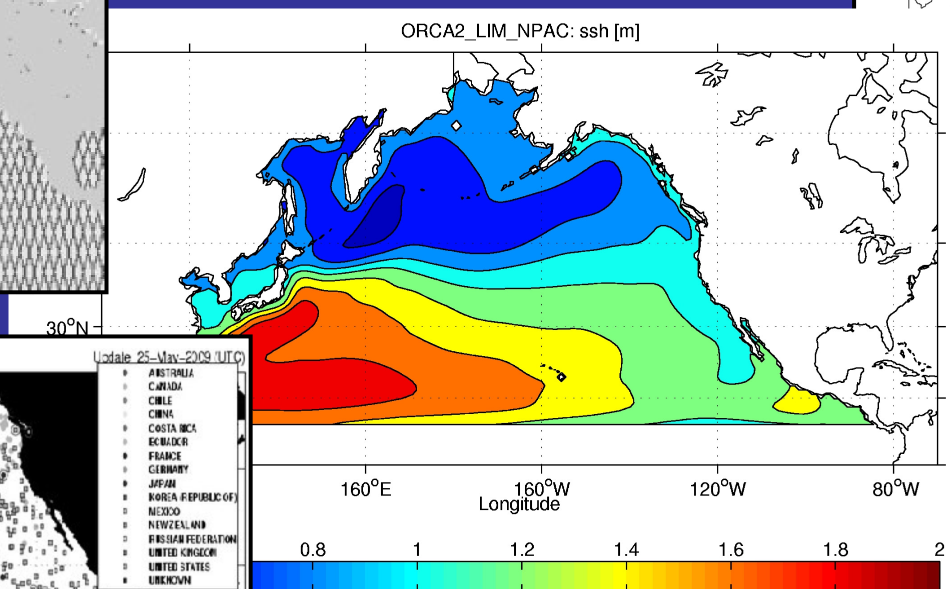
$$\mathbf{Hc} = \mathbf{y}$$

data

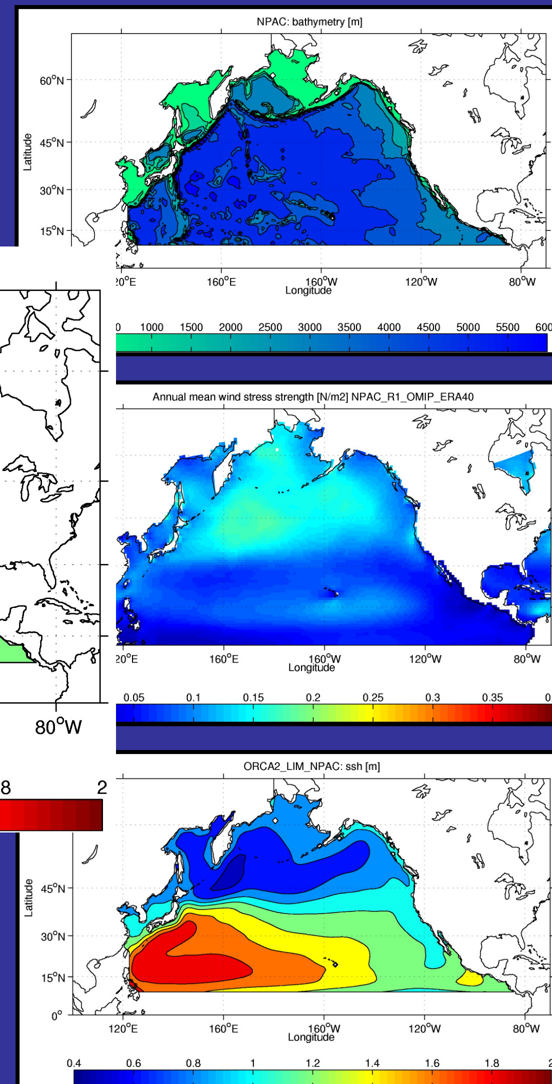




model



control

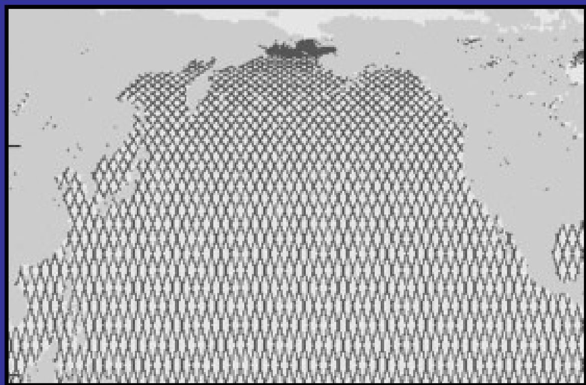


Typical case:

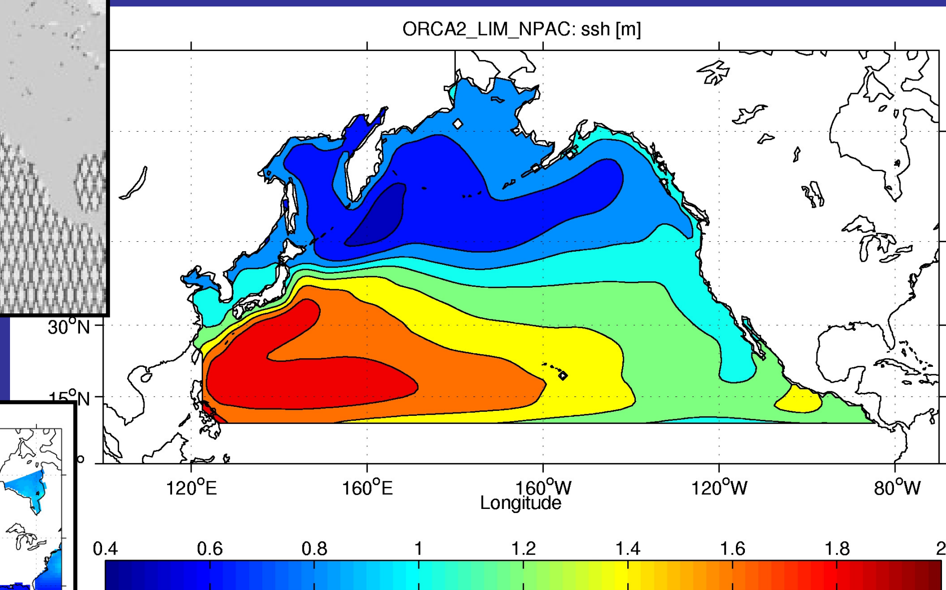
M << N

Background

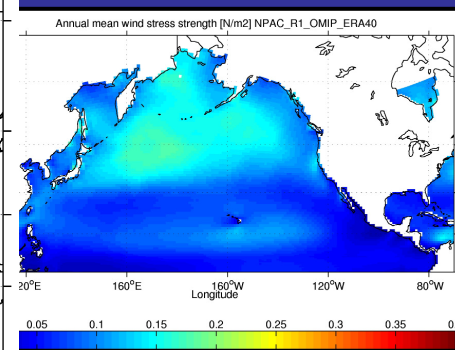
data



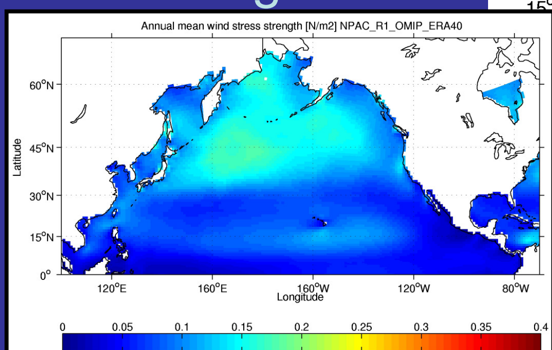
model



control



background



$$\mathbf{H}\mathbf{c} = \mathbf{y} + \delta\mathbf{y}, \quad \mathbf{c} = \mathbf{c}^b + \delta\mathbf{c}$$

M+N > N

- Background (estimation)
- Least squares solution
- Error in data and background

4D-Var

cost function (weighted least squares problem):

$$J = \delta\mathbf{c}^{\mathrm{T}}\mathbf{Q}_C^{-1}\delta\mathbf{c} + \underbrace{(\tilde{\mathbf{H}}\delta\mathbf{c}-\mathbf{d})^{\mathrm{T}}}_{\delta\mathbf{y}^{\mathrm{T}}}\mathbf{R}^{-1}\underbrace{(\tilde{\mathbf{H}}\delta\mathbf{c}-\mathbf{d})}_{\delta\mathbf{y}}, \quad \mathbf{d} = \mathbf{y} - \tilde{\mathbf{H}}\mathbf{c}$$

error covariance models (weights):

$$\mathbf{Q}_C = \left\langle \delta\mathbf{c}\,\delta\mathbf{c}^{\mathrm{T}} \right\rangle, \quad \mathbf{R} = \left\langle \delta\mathbf{y}\,\delta\mathbf{y}^{\mathrm{T}} \right\rangle$$

exact solution:

$$\delta\mathbf{c}^a = \mathbf{Q}_C\tilde{\mathbf{H}}^{\mathrm{T}}(\tilde{\mathbf{H}}\mathbf{Q}_C\tilde{\mathbf{H}}^{\mathrm{T}} + \mathbf{R})^{-1}\mathbf{d}$$

optimal solution is contaminated by measurement error:

$$\delta\mathbf{c}^a = \delta\mathbf{c}^a_{SIGNAL} + \delta\mathbf{c}^a_{NOISE}$$

How to separate signal and noise?

Observability Matrix in 4D-Var (Johnson et al., QJRMS 2006)

reformulation of the exact solution:

$$\delta\mathbf{c}^a = \mathbf{Q}_C^{1/2}\mathbf{D}^{\mathrm{T}}(\mathbf{D}\mathbf{D}^{\mathrm{T}} + \mathbf{I}_M)^{-1}\hat{\mathbf{d}}, \quad \hat{\mathbf{d}} = \mathbf{R}^{-1/2}\mathbf{d}$$

$$\mathbf{D} = \mathbf{R}^{-1/2}\tilde{\mathbf{H}}\mathbf{Q}_C^{1/2} : \text{observability matrix}$$

further reformulation using SVD of D:

$$\mathbf{D} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^{\mathrm{T}}, \quad \mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \cdots, \lambda_M\}$$

$$\delta\mathbf{c}^a = \mathbf{Q}_C^{1/2}\sum_{m=1}^{M}\beta_m\mathbf{v}_m, \quad \beta_m = \frac{\lambda_m}{\lambda_m^2+1}\mathbf{u}_m^{\mathrm{T}}\hat{\mathbf{d}}$$

separation of signal and noise:

$$\delta\mathbf{c}^a = \delta\mathbf{c}^a_{SIGNAL} + \delta\mathbf{c}^a_{NOISE}, \quad \begin{cases} \delta\mathbf{c}^a_{SIGNAL} = \mathbf{Q}_C^{1/2}\sum_{m=1}^{r}\beta_m\mathbf{v}_m & \text{for} \quad \lambda_m^2 > 1 \\ \delta\mathbf{c}^a_{NOISE} = \mathbf{Q}_C^{1/2}\sum_{m=r}^{M}\beta_m\mathbf{v}_m & \text{for} \quad \lambda_m^2 < 1 \end{cases}$$

Algorithm

Q. How to compute SVD of D for large problem (OGCM)?:

D is MxN matrix (too large to perform SVD for OGCM):

$$\mathbf{D} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{T}}, \quad \boldsymbol{\Lambda} = \mathrm{diag}\left\{\lambda_1, \cdots, \lambda_M\right\}$$

and complicated (you can not write down M matrix for OGCM):

$$\mathbf{D} = \mathbf{R}^{-1/2}\mathbf{H}\mathbf{M}\mathbf{Q}_C^{1/2}$$

A. Solve the problem on a data space using adjoint / forward models:

→ DD^T is MxM matrix (small enough to perform SVD)

$$\mathbf{D}\mathbf{D}^{\mathrm{T}} = \mathbf{U}\boldsymbol{\Lambda}^2\mathbf{U}^{\mathrm{T}}, \quad \boldsymbol{\Lambda}^2 = \mathrm{diag}\left\{\lambda_1^2, \cdots, \lambda_M^2\right\}$$

→ DD^T can be constructed indirectly column by column

$$\hat{\mathbf{s}}_m = \mathbf{D}\mathbf{D}^{\mathrm{T}}\mathbf{e}_m = \mathbf{R}^{-1/2}\mathbf{H}\mathbf{M}\mathbf{Q}_C\mathbf{M}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{R}^{-1/2}\mathbf{e}_m$$

Algorithm

1.construct DD^T column by column:

$$\boldsymbol{\alpha}_m = \mathbf{M}^T\mathbf{H}^T\mathbf{R}^{-1/2}\mathbf{e}_m \quad \text{: adjoint equation}$$

$$\mathbf{x}_m = \mathbf{M}\mathbf{Q}_C\boldsymbol{\alpha}_m \quad \text{: forward equation}$$

$$\hat{\mathbf{s}}_m = \mathbf{R}^{-1/2}\mathbf{H}\mathbf{x}_m \quad \text{: normalized measurement}$$

2.perform SVD of DD^T:

$$\mathbf{D}\mathbf{D}^T = [\hat{\mathbf{s}}_1 \cdots \hat{\mathbf{s}}_M] = \mathbf{U}\boldsymbol{\Lambda}^2\mathbf{U}^T$$

3.construct the right singular vector:

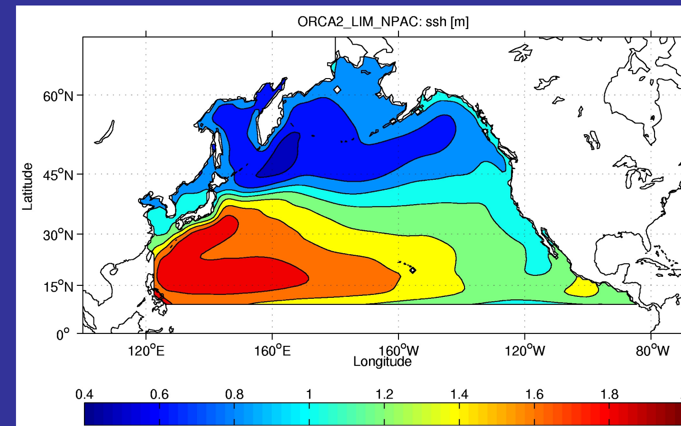
$$\boldsymbol{\gamma}_m = \mathbf{M}^T\mathbf{H}^T\mathbf{R}^{-1/2}\lambda_m^{-1}\mathbf{u}_m \quad \text{: adjoint equation}$$

$$\mathbf{v}_m = \mathbf{Q}_C^{1/2}\boldsymbol{\gamma}_m \quad \text{: convolution}$$

4.compute δc^a_{SIGNAL}:

$$\delta\mathbf{c}^a_{SIGNAL} = \mathbf{Q}_C^{1/2}\sum_{m=1}^{r}\beta_m\mathbf{v}_m \quad \text{for} \quad \lambda_m^2 > 1$$

Test of the algorithm (up to step 2)



quasi-geostrophic model:

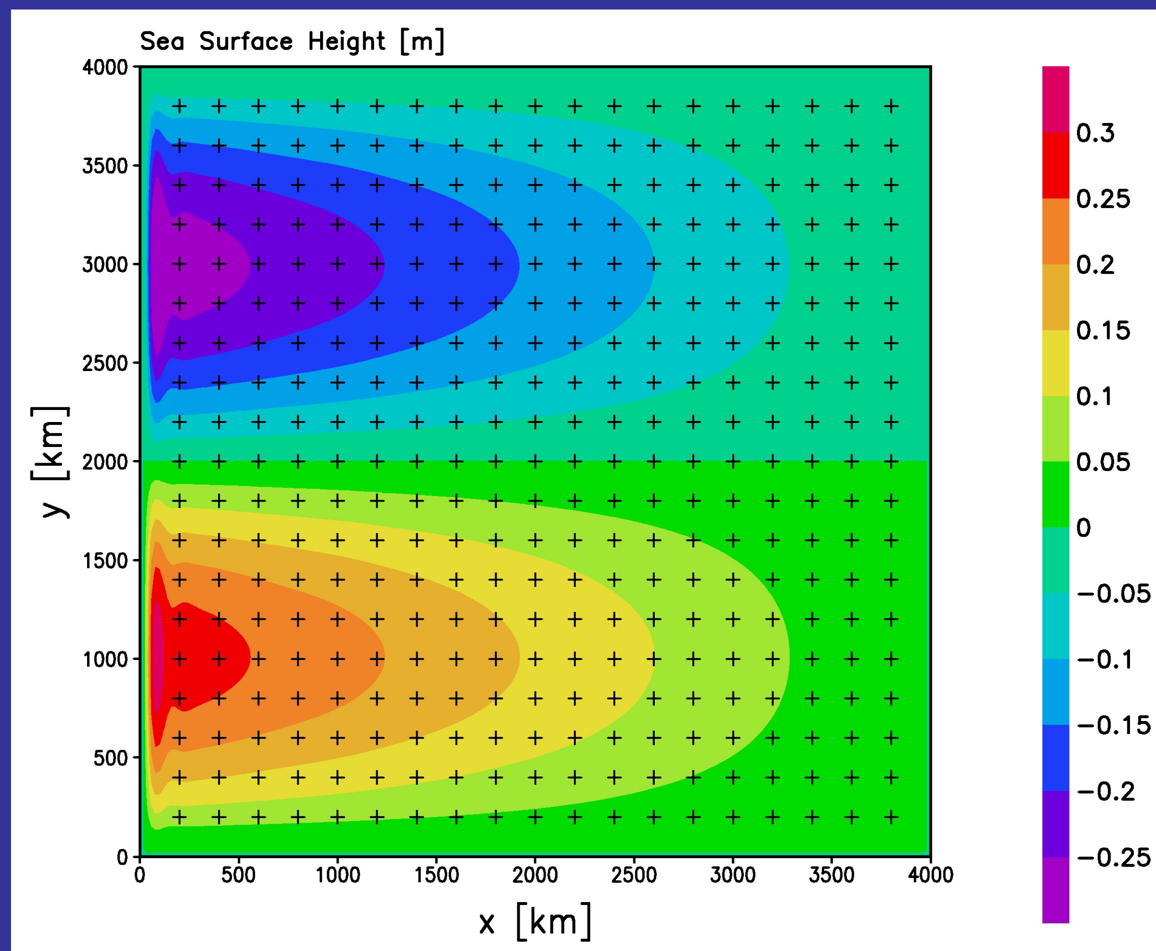
- 1.5 layer
- linear
- beta plane
- steady wind stress
- 20km x 20km

observation:

- sea surface height at 200km x 200km mesh at day 10 (M=361)

control variables:

- initial potential vorticity (N=40000)



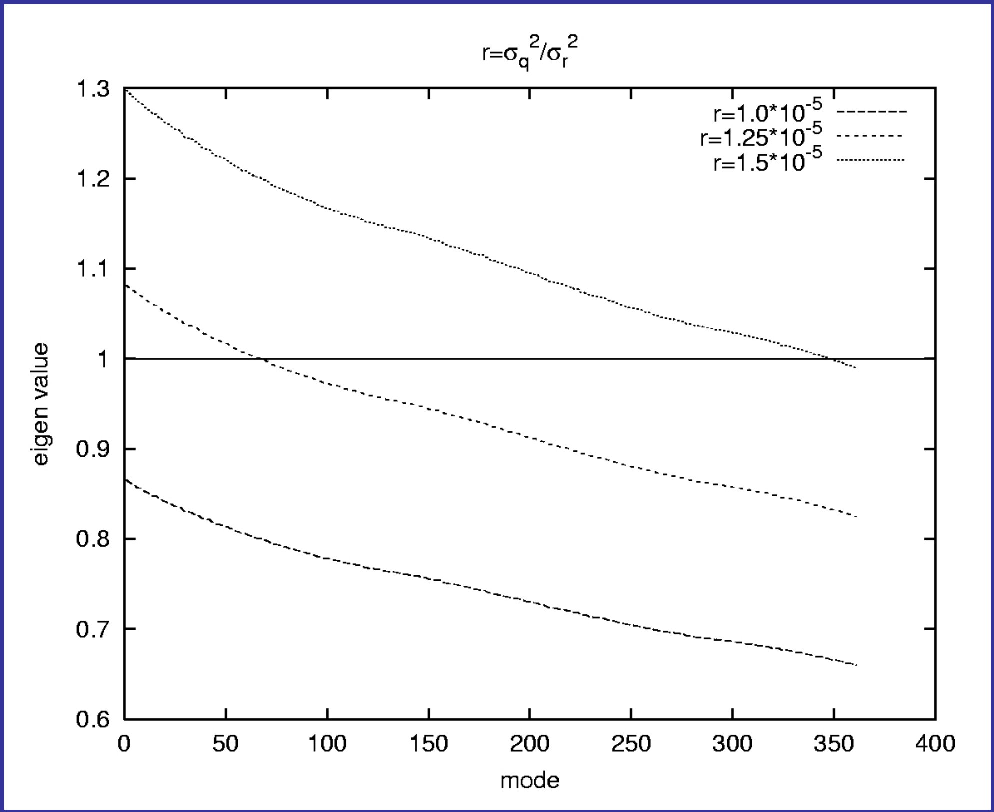
Test of the algorithm (preliminary)

Control variables: initial potential vorticity

Error covariance models:

$$\mathbf{Q}_C = \sigma_C^2 \mathbf{I}_N, \quad \mathbf{R} = \sigma_R^2 \mathbf{I}_M$$

$$\mathbf{D}\mathbf{D}^{\mathrm{T}} = \mathbf{U} \frac{\sigma_C^2}{\sigma_R^2} \mathbf{\Sigma}^2 \mathbf{U}^{\mathrm{T}}$$



top: almost all modes are above noise

middle: the first 50 modes are above noise

bottom: entire solution is meaningless

Summary

- the optimal estimation of a control variables in a 4D-Var analysis can be divided to
  1. signal (observable mode)

  and 2. noise (unobservable mode).
- the algorithm to compute singular values and vectors of the observability matrix are derived.
- (half of) the algorithm was tested using QG model.

Future work

- implement the algorithm on OPA model (OPAVAR)
- perform statistical test of the error covariance model against data

Acknowledgement

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